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Nonlinear breakup of an asymmetric electrohydrodynamic jet

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Abstract. Nonlinear asymmetric breakup of a circular capillary jet stressed at the surface by an electrical field is presented. The method of straining of coordinates is used to calculate the cutoff wave number. The numerical results show that an initially sinusoidal wave develops in an asymmetrical form by the nonlinear effect from higher harmonics. It is shown that a linear relation-ship exists between the breakup times and the logarithms of the amplitudes of the wave. The slopes of these lines can be used to calculate the actual growth rates. The effective growth rates are displayed in a graphical form.

PACS. 47.65.+a Electrohydrodynamics

1 Introduction

The stability of a liquid jet of circular cross-section under a small external disturbance at the surface of the jet has been considered by many investigators. Work on this subject was started in the nineteenth century by Bidone, Savart, and others. Based on the linearized theory, Lord Rayleigh [1] gave a detailed analytical explanation of this phenomenon. Experimental studies [2] of the breakup of a liquid jet have shown that Rayleigh's uniform drop model is inadequate. In these experiments, non-sinusoidal deformations were observed indicating nonlinear behavior. These observations motivated several researchers to formulate a nonlinear theory for the breakup of liquid jets.

Yuen [3] developed a third-order nonlinear theory for this problem using the method of straining coordinates. Wang [4], Nayfeh [5], Chaudhary and Redekopp [6], Kakutani *et al.* [7], and Lafrance [8] have also carried out nonlinear analyses of the problem. In more recent years, electrohydrodynamic stability of capillary jets has been investigated by Gañán-Calvo [9] and Mestel [10].

In recent years, there has been considerable interest in the effect of an electrical or a magnetic field on the stability of a fluid jet. The interest primarily arose from astrophysical problems, but more recently the question has been of growing importance for technological devices. The physical properties of liquid jets play a fundamental role in a number of applications such as spray drying, electronic ink jet printing, the spinning of synthetic fibers, and fuel atomization.

The theoretical and the experimental studies which are concerned with the effect of electrical fields on the capillary waves of a circular jet were investigated by Melcher [11]. However, the analysis of this study was confined within the frame work of the linear theory. Malik and Singh [12] carried out the nonlinear analysis of the same problem by using the method of straining coordinates. Lardner and Trehan [13] examined the effect of a magnetic field on the nonlinear capillary instability of a fluid jet using method multiple scales, while, Trehan and Lardner [15] investigated the nonlinear standing waves of the same problem.

For an astrophysical problem, Chhabra and Trehan [15] examined the effect of a uniform axial magnetic field on the nonlinear instability of a self-gravitating infinite cylinder. All these investigations, however, concerned with axisymmetric circular columnar fluids, but asymmetric nonlinear analysis of circular jets does not appear to be investigated. Although there are some reports on nonaxisymmetric breakup of a liquid jet, *e.g.*, Yang [16], these analyses are confined within the frame work of linear theory.

The purpose of this paper is to investigate the nonaxisymmetric nonlinear breakup of laminar conducting liquid jet in the presence of an electric field. In this presentation, a nonlinear problem is considered, in which a jet of fluid having a circular cylindrical geometry is stressed at the surface by a radial electric field. In this paper, by the method of straining of coordinates, we have developed a third order asymmetric nonlinear theory on the propagation of waves over the surface of circular jet. A solution by the multiple scales method to the same problem has been obtained by Lee [17] recently. The basic equations with the accompanying boundary conditions are given in Section 2. The first order theory and the linear dispersion relation are obtained in Section 3. In Section 4 we have derived second order solutions. In Section 5 third order problem is considered and the cut-off wave number is obtained. Finally Section 6 is devoted to some numerical examples.

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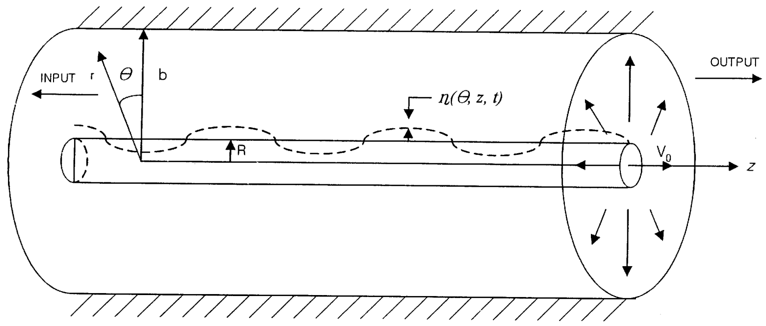

Fig. 1. Configuration for the cylindrical electric jet showing the outer electrode at $r = b$ coxial with the equilibrium jet.

2 Formulation

We consider an incompressible, inviscid fluid jet whose density is ρ and whose radius is R stressed at the surface by constant radial field. The fluid is highly conducted so as to exclude contribution of the electric field from the fluid in the jet. The radius b of outer conducting cylinder is much larger than the perturbations wavelength of the jet. We use the cylindrical polar coordinates (r, θ, z) with z-axis taken along the axis of the jet. The electric field is applied by means of external rigid, perfectly conducting parallel electrode at $r = b$. The cylinder is at a fixed potential V_0 relative to that of the jet. The interface is defined by a function of θ, z and time. Let $\eta(\theta, z, t)$ denote the elevation of the free surface measured from the unperturbed level (see Fig. 1).

The consequential dynamics are retained if regions, external to the jet, are assumed to be filled with a fluid of small density (compared to the density of the fluid in the jet), and the effects of gravity are ignored. Now, a periodic initial disturbance is given at the surface of the jet. The motion is assumed to be irrotational. If $\mathbf{u}$ and $\mathbf{E}$ denote velocity field and the electric field, respectively, at any time t, then

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{E} = 0.$$

If ϕ and ψ denote velocity and electric potential, respectively, so that $\mathbf{u} = \nabla\phi$, and $\mathbf{E} = -\nabla\psi$, then the equations for ϕ and ψ are given by

$$\nabla^2\phi = 0, \tag{2.1}$$

for $r \leq R + \eta(\theta, z, t)$,

$$\nabla^2\psi = 0, \tag{2.2}$$

for $r \geq R + \eta(\theta, z, t)$.

The unit normal $\mathbf{n}$ to the surface is given by

$$\begin{aligned}\mathbf{n} &= \frac{\nabla F}{|\nabla F|} \\ &= \left(\mathbf{e}_r - \frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_\theta - \frac{\partial \eta}{\partial z}\mathbf{e}_z\right)\left\{1+\left(\frac{\partial \eta}{r\partial \theta}\right)^2+\left(\frac{\partial \eta}{\partial z}\right)^2\right\}^{-\frac{1}{2}},\end{aligned} \tag{2.3}$$

where $F = 0$ is the equation of the surface of jet. The condition that the electric field is satisfied on the deformed surface of the jet and at external boundaries is

$$\boldsymbol{n} \times [\![\boldsymbol{E}]\!] = 0 \quad \text{at} \quad r = R + \eta \quad \text{and} \quad r = b, \tag{2.4}$$

where $[\![\ \cdot]\!]$ represents the jump across the surface of the jet.

All physical variables are normalized by using the radius of the undisturbed jet R for the characteristic length, and $\sqrt{T/\rho R}$ for the characteristic speed, and RE_0^2/T for the characteristic electric field parameter. Here, T is the surface tension and E_0 is the field strength at the surface of the undeformed jet. In the following, the primes on the dimensionless variable $\eta' = \eta/R, b' = b/R, d' = d/R$, and so on, are omitted for conciseness.

The condition that the interface is moving with the fluid leads to

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial r} = -\frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} - \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z} \quad \text{at} \quad r = 1 + \eta. \tag{2.5}$$

Now the boundary condition at the free surface is,

$$\begin{aligned}&-\frac{\partial \phi}{\partial t} - \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\} \\ &\quad - \frac{1}{r|\nabla F|}\left\{1 + \left(\frac{1}{r}\frac{\partial \eta}{\partial \theta}\right)^2 \frac{2}{|\nabla F|^2}\right\} \\ &\quad + \frac{1}{|\nabla F|^3}\left[\frac{\partial^2 \eta}{\partial z^2}\left\{1 + \left(\frac{1}{r}\frac{\partial \eta}{\partial \theta}\right)^2\right\}\right. \\ &\quad \left. - \frac{2}{r^2}\frac{\partial \eta}{\partial \theta}\frac{\partial^2 \eta}{\partial \theta \partial z}\frac{\partial \eta}{\partial z} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}\left\{1 + \left(\frac{\partial \eta}{\partial z}\right)^2\right\}\right] \\ &\quad + \Gamma\left[(\nabla\psi \cdot \boldsymbol{n})^2 - \frac{1}{2}|\nabla\psi|^2\right] = C,\end{aligned} \tag{2.6}$$

where

$$\Gamma = \frac{E_0^2 R}{4\pi T} = \frac{V_0^2}{4\pi T R},$$

and F is given by

$$F = r - \eta(\theta, z, t) - 1.$$

Since the fluid is infinitely conducting (2.4), can now be written as

$$\psi = 0 \quad \text{at } r = 1 + \eta(\theta, z, t). \tag{2.7}$$

At time $t = 0$, disturbance of amplitude η_0 and wave number K is imposed on the surface $r = 1$ of the jet. We assume the initial conditions to be

$$\eta(z,0) = \eta_0 \cos Kz \cos n\theta + \eta_0^2\left(-\frac{1}{8}\right) + \eta_0^4\left(-\frac{1}{128}\right), \tag{2.8}$$

$$\frac{\partial \eta(z,0)}{\partial t} = 0. \tag{2.9}$$

For an initially sinusoidal disturbance with the undisturbed radius equal to 1, we have

$$r = R + \eta_0 \cos Kz \cos n\theta. \tag{2.10}$$

For conservation of mass of a column of liquid of length π/k, and volume π^2/k, we have

$$\begin{aligned}\frac{\pi^2}{K} &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/K}(R + \eta_0 \cos Kz \cos n\theta)^2 \mathrm{d}z\mathrm{d}\theta \\ &= R^2\frac{\pi^2}{K} + \frac{\eta_0^2}{4}\frac{\pi^2}{K}\cdot\end{aligned}$$

Therefore

$$R = \left(1 - \frac{\eta_0^2}{4}\right)^{1/2} = 1 + \eta_0^2(-\frac{1}{8}) + \eta_0^4(-\frac{1}{128}) + \ldots.$$

Substituting this back into (2.10), we obtain (2.8).

The nonlinear stability problem posed by (2.1–2.2) and (2.5–2.9) is examined by the method of the strained coordinates [18]. We now assume that the surface disturbance and the potential functions can be expanded in a perturbation series in terms of smallness parameter η_0. Thus, we write

$$\phi(r,\theta,z,t) = \sum_{n=1}^{\infty}\eta_0^n\phi_n(r,\theta,z,t), \tag{2.11}$$

$$\psi(r,\theta,z,t) = \sum_{n=0}^{\infty}\eta_0^n\psi_n(r,\theta,z,t), \tag{2.12}$$

and

$$\eta(\theta,z,t) = \sum_{n=1}^{\infty}\eta_0^n\eta_n(\theta,z,t). \tag{2.13}$$

Now, we use following strained coordinates:

$$\tau = t\nu = t\left[\sum_{n=1}^{\infty}\nu_n\eta_0^{n-1}\right], \tag{2.14}$$

$$\xi = k_c z = z\left[\sum_{n=1}^{\infty}k_n\eta_0^{n-1}\right], \tag{2.15}$$

and we introduce

$$\mathcal{K} = \frac{K}{(k_1 + \eta_0 k_2 + \eta_0^2 k_3 + \cdots)}\cdot \tag{2.16}$$

If we substitute (2.11–2.13) into (2.5–2.9), boundary conditions for various orders are obtained. A Maclaurin series expansion of the boundary conditions at $r = 1$ provides successive orders of approximation to these conditions which are then used to specify the problem in those orders.

3 Linear theory

Since (2.1) and (2.2) are linear, they must be satisfied by each of the ϕ_m and ψ_m separately. The corresponding boundary conditions and initial conditions are obtained by substituting the expressions (2.11, 2.12) and (2.13) for ϕ, ψ and η, respectively into (2.5–2.9), and equating equal powers of η_0^n. Equating the coefficient of the first power of η_0 leads to

$$\nabla_0^2\phi_1 = 0, \tag{3.1}$$

$$\nabla_0^2\psi_1 = 0, \tag{3.2}$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + k_1^2\frac{\partial^2}{\partial\xi^2}\cdot$$

The various boundary conditions at the interface are (at $r = 1$)

$$-\nu_1\frac{\partial\eta_1}{\partial\tau} + \frac{\partial\phi_1}{\partial r} = 0, \tag{3.3}$$

$$\psi_1 + \eta_1\frac{\partial\psi_0}{\partial r} = 0, \tag{3.4}$$

$$\begin{aligned}-\nu_1\frac{\partial\phi_1}{\partial\tau} + \eta_1 + \frac{\partial^2\eta_1}{\partial\theta^2} + k_1^2\frac{\partial^2\eta_1}{\partial\xi^2} \\ + \mathit{\Gamma}\left(\frac{\partial\psi_1}{\partial r}\frac{\partial\psi_0}{\partial r} + \eta_1\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_0}{\partial r^2}\right) = 0,\end{aligned} \tag{3.5}$$

where $\psi_0 = -\ln r$.

We can set $\nu_1 = 1$, and solutions to (3.3) and (3.4) are given by

$$\eta_1 = \cosh\omega_1\tau\cos\mathcal{K}\xi\cos n\theta, \tag{3.6}$$

$$\phi_1 = \frac{\omega}{k}\frac{I_n(kr)}{I_n'(k)}\sinh\omega_1\tau\cos\mathcal{K}\xi\cos n\theta, \tag{3.7}$$

$$\psi_1 = \frac{K_n(kr)}{K_n(k)}\cosh\omega_1\tau\cos\mathcal{K}\xi\cos n\theta, \tag{3.8}$$

where $k = k_1\mathcal{K}$ and k_1 is from the subsequent equation (3.11) and $I_n(kr), K_n(kr)$ are the modified Bessel functions of the first and second kind, respectively. Substituting (3.6–3.8) into (3.5), we obtain following dispersion relation

$$\omega^2 = \frac{k}{I_a}\left\{1 - k^2 - n^2 - \mathit{\Gamma}(1 + kK_a)\right\}, \tag{3.9}$$

where

$$I_a = \frac{I_n(k)}{I_n'(k)}, \qquad K_a = \frac{K_n'(k)}{K_n(k)}\cdot \tag{3.10}$$

We choose k_1 to be the solution of the equation

$$1 - k_1^2 - n^2 - \mathit{\Gamma}\left\{1 + k_1\frac{K_n'(k_1)}{K_n(k_1)}\right\} = 0. \tag{3.11}$$

For $k > k_1$, the solutions are stable, and if $k < k_1$, the surface waves grow. We see that k_1 is the linear cut off wave number which separates the stable from unstable disturbances. It is well-known that in the linear theory asymmetric disturbances are always stable if no electric field is present. Equations (3.6–3.9) agree with those given by Melcher [11].

4 Second order solutions

The second order problem is governed by

$$\nabla_0^2\phi_2 = -2k_1k_2\frac{\partial^2\phi_1}{\partial\xi^2}, \tag{4.1}$$

$$\nabla_0^2\psi_2 = -2k_1k_2\frac{\partial^2\psi_1}{\partial\xi^2}\cdot \tag{4.2}$$

The boundary conditions at $r = 1$ are:

$$L(\eta_2,\phi_2) = \nu_2\frac{\partial\eta_1}{\partial\tau} - \frac{\partial^2\phi_1}{\partial r^2}\eta_1 + k_1^2\frac{\partial\eta_1}{\partial\xi}\frac{\partial\phi_1}{\partial\xi} + \frac{\partial\eta_1}{\partial\theta}\frac{\partial\phi_1}{\partial\theta}, \tag{4.3}$$

$$M(\eta_2,\psi_2) = -\frac{\partial\psi_1}{\partial r}\eta_1 - \frac{\partial^2\psi_0}{\partial r^2}\frac{\eta_1^2}{2}, \tag{4.4}$$

$$\begin{aligned}N(\eta_2,\phi_2,\psi_2) &= \nu_2\frac{\partial\phi_1}{\partial\tau} + \nu_1\frac{\partial^2\phi_1}{\partial\tau\partial r}\eta_1 \\ &+ \frac{1}{2}\left\{\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \left(\frac{\partial\phi_1}{\partial\theta}\right)^2 + k_1^2\left(\frac{\partial\phi_1}{\partial\xi}\right)^2\right\} \\ &- 2k_1k_2\frac{\partial^2\eta_1}{\partial\xi^2} + \eta_1^2 - \frac{1}{2}k_1^2\left(\frac{\partial\eta_1}{\partial\xi}\right)^2 + \frac{3}{2}\left(\frac{\partial\eta_1}{\partial\theta}\right)^2 + 2\eta_1\frac{\partial^2\eta_1}{\partial\theta^2} \\ &- \Gamma\left[\frac{1}{2}\left(\frac{\partial\psi_1}{\partial r}\right)^2 + \eta_1\frac{\partial\psi_1}{\partial r}\frac{\partial^2\psi_0}{\partial r^2} + \eta_1\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_1}{\partial r^2}\right. \\ &+ \frac{\eta_1^2}{2}\left\{\left(\frac{\partial^2\psi_0}{\partial r^2}\right)^2 + \frac{\partial\psi_0}{\partial r}\frac{\partial^3\psi_0}{\partial r^3}\right\} \\ &- 2\frac{\partial\psi_0}{\partial r}\left(k_1^2\frac{\partial\psi_1}{\partial\xi}\frac{\partial\eta_1}{\partial\xi} + \frac{\partial\psi_1}{\partial\theta}\frac{\partial\eta_1}{\partial\theta}\right) \\ &- \left(\frac{\partial\psi_0}{\partial r}\right)^2\left\{k_1^2\left(\frac{\partial\eta_1}{\partial\xi}\right)^2 + \left(\frac{\partial\eta_1}{\partial\theta}\right)^2\right\} \\ &\left. - \frac{1}{2}\left\{k_1^2\left(\frac{\partial\psi_1}{\partial\xi}\right)^2 + \left(\frac{\partial\psi_1}{\partial\theta}\right)^2\right\}\right], \end{aligned} \tag{4.5}$$

where $L(\eta_2,\phi_2), M(\eta_2,\psi_2)$ and $N(\eta_2,\phi_2,\psi_2)$ denote the left-hand sides of (3.3, 3.4) and (3.5) with η_1, ϕ_1 and ψ_1 being replaced by η_2, ϕ_2 and ψ_2, respectively.

The initial conditions for the second order problem are

$$\eta_2(\theta,\xi,0) = -\frac{1}{8}, \quad \frac{\partial\eta_2}{\partial\tau} = 0. \tag{4.6}$$

For the solution of the second order problem, we assume

$$\eta_2(\xi,\theta,\tau) = B_{21}(\tau)\cos 2\mathcal{K}\xi\cos 2n\theta + B_{22}(\tau)\cos 2\mathcal{K}\xi + B_{23}(\tau)\cos 2n\theta + D_2(\tau), \tag{4.7}$$

Using (4.7) and first order solutions in (4.1–4.6), we obtain the following second order solutions:

$$B_{21}(\tau) = a_{21}\cosh\omega_{21}\tau + b_{21}\cosh 2\omega_1\tau + c_{21}, \tag{4.8a}$$
$$B_{22}(\tau) = a_{22}\cosh\omega_{22}\tau + b_{22}\cosh 2\omega_1\tau + c_{22}, \tag{4.8b}$$
$$B_{23}(\tau) = a_{23}\cosh\omega_{23}\tau + b_{23}\cosh 2\omega_1\tau + c_{23}, \tag{4.8c}$$

$$\begin{aligned}\psi_2(r,\theta,\xi,\tau) &= \{B_{21}(\tau)+S_{22}(\tau)\}\frac{K_{2n}(2kr)}{K_{2n}(2k)}\cos 2\mathcal{K}\xi\cos 2n\theta \\ &+ \{B_{22}(\tau)+S_{22}(\tau)\}\frac{K_0(2kr)}{K_0(2k)}\cos 2\mathcal{K}\xi \\ &+ \{B_{23}(\tau)+S_{22}(\tau)\}\frac{\cos 2n\theta}{r^{2n}} + E_2(\tau), \end{aligned} \tag{4.9}$$

$$\begin{aligned}\phi_2(r,\theta,\xi,\tau) &= \left\{\frac{\partial B_{21}(\tau)}{\partial\tau} + P_{21}\sinh 2\omega_1\tau\right\}\frac{I_{2n}(2kr)}{2kI'_{2n}(2k)} \\ &\times\cos 2\mathcal{K}\xi\cos 2n\theta \\ &+ \left\{\frac{\partial B_{22}(\tau)}{\partial\tau} + P_{22}\sinh 2\omega_1\tau\right\}\frac{I_0(2kr)}{2kI_1(2k)}\cos 2\mathcal{K}\xi \\ &+ \left\{\frac{\partial B_{23}(\tau)}{\partial\tau} + P_{23}\sinh 2\omega_1\tau\right\}\frac{r^{2n}}{2n}\cos 2n\theta + F_2(\tau), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned}S_{22}(\tau) &= -\frac{2kK_a+1}{16}\{\cosh(2\omega_1\tau)+1\} \\ D_2(\tau) &= -\frac{\cosh(2\omega_1\tau)+1}{16} \\ E_2(\tau) &= -\frac{kK_a+1}{8}\{\cosh(2\omega_1\tau)+1\} \\ F_2(\tau) &= -\frac{\tau}{16}[\omega_1^2\{1-I_a^2(1+n^2/k^2)\} \\ &\quad + 3 - k^2 - n^2 - \Gamma\Delta_4] \\ &\quad - \frac{\sinh(2\omega_1\tau)}{32\omega_1}[\omega_1^2\{3+I_a^2(1+n^2/k^2)\} \\ &\quad + 3 - k^2 - n^2 - \Gamma\Delta_4], \\ P_{21} &= \frac{1}{8}\omega_1\{1-2kI_a(1+n^2/k^2)\}, \\ P_{22} &= \frac{1}{8}\omega_1(1-2kI_a), \\ P_{23} &= \frac{1}{8}\omega_1(1-2I_an^2/k), \end{aligned}$$

and a_{21} and so on are listed in the Appendix A. If we set n and Γ equal to zero in (4.7–4.10), these equations completely agree with those obtained by Yuen except two typographical omissions which were also pointed out by Rutland and Jameson [19].

5 Third order problem

We use the first and second order solutions to derive the solution of the third order problem. In this section we calculate k_3 and ν_3.

$$\nabla_0^2\phi_3 = 2k_1k_3\mathcal{K}^2\omega_1\frac{I_n(kr)}{I_n'(k)}\sinh\omega_1\tau\cos\mathcal{K}\xi\cos n\theta, \quad (5.1)$$

$$\nabla_0^2\psi_3 = 2k_1k_3\mathcal{K}^2\frac{K_n(kr)}{K_n(k)}\cosh\omega_1\tau\cos\mathcal{K}\xi\cos n\theta. \quad (5.2)$$

The boundary conditions at $r=1$ are as follows

$$-\frac{\partial\eta_3}{\partial\tau}+\frac{\partial\phi_3}{\partial r} = P_{31}(\tau)\cos\mathcal{K}\xi\cos n\theta+\cdots, \quad (5.3)$$

$$\psi_3+\eta_3\frac{\partial\psi_0}{\partial r} = R_{31}(\tau)\cos\mathcal{K}\xi\cos n\theta+\cdots, \quad (5.4)$$

$$-\frac{\partial\phi_3}{\partial\tau}+\left(1+\frac{\partial^2}{\partial\theta^2}+k_1^2\frac{\partial^2}{\partial\xi^2}\right)\eta_3+\Gamma\frac{\partial\psi_0}{\partial r}\left(\frac{\partial\psi_3}{\partial r}+\eta_3\frac{\partial^2\psi_0}{\partial r^2}\right) = [Q_{31}(\tau)-\Gamma S_{31}(\tau)]\cos\mathcal{K}\xi\cos n\theta+\cdots, \quad (5.5)$$

and the initial conditions are

$$\eta_3(\theta,\xi,0)=0, \quad \frac{\partial\eta_3(\theta,\xi,0)}{\partial\tau}=0, \quad (5.6)$$

where the expressions for P, Q, R and S are given in the Appendix B. Following the same approach as in Section 4, we assume

$$\eta_3(\tau) = B_{31}(\tau)\cos\mathcal{K}\xi\cos n\theta+\cdots. \quad (5.7)$$

The third order electric and velocity potentials are now

$$\psi_3 = \left[\{B_{31}(\tau)+R_{31}(\tau)\}\frac{K_n(kr)}{K_n(k)}+\left\{-\frac{k_3}{k_1}kr\frac{K_{n+1}(kr)}{K_n(k)}+\frac{k_3}{k_1}k\frac{K_{n+1}(k)}{K_n(k)}\frac{K_n(kr)}{K_n(k)}\right\}\times\cosh\omega_1\tau\right]\cos\mathcal{K}\xi\cos n\theta+\cdots, \quad (5.8)$$

$$\phi_3 = \left[\left\{\frac{\partial B_{31}(\tau)}{\partial\tau}+P_{31}(\tau)-\frac{k_3}{k_1}\omega_1\left(-n+\frac{n^2+k^2}{k}I_a\right)\times\sinh\omega_1\tau\right\}\frac{I_n(kr)}{kI_n'(k)}+\frac{k_3}{k_1}\omega_1 r\frac{I_{n+1}(kr)}{kI_n'(k)}\sinh\omega_1\tau\right]\times\cos\mathcal{K}\xi\cos n\theta+\cdots. \quad (5.9)$$

Substituting from (5.8) and (5.9) into (5.5), from the coefficients of $\cos\mathcal{K}\xi\cos n\theta$, we obtain the following differential equation to determine $B_{31}(\tau)$:

$$\frac{\partial^2 B_{31}(\tau)}{\partial\tau^2}-\omega_1^2B_{31}(\tau) = -\frac{k}{I_a}\left[\omega_1^2p_{31}\frac{I_a}{k}+q_{31}-\Gamma(s_{31}-kK_ar_{31})-\frac{k_3}{k_1}\left\{\omega_1^2\left(\frac{n^2+k^2}{k^2}I_a^2-1\right)-\Gamma(k^2+n^2-k^2K_a^2)\right\}\right]\cosh\omega_1\tau. \quad (5.10)$$

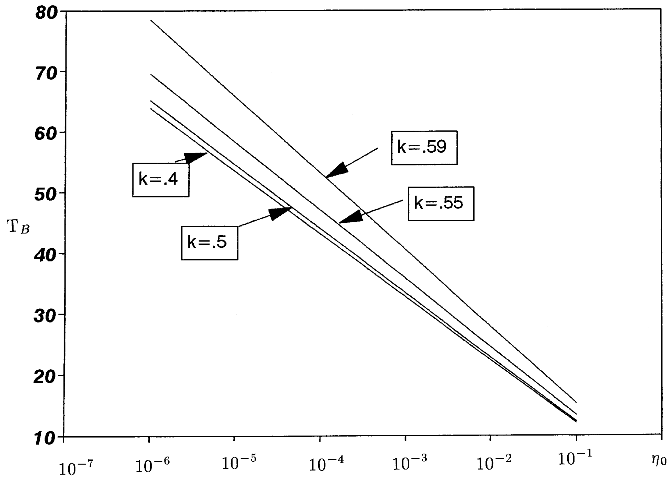

Fig. 2. Exponential character of the growth of surfaces waves ($n=1, \Gamma=1$).

In order η_3 to be stable for $K > k_1$, the right-hand side of (5.10) has to be set equal to be zero. This determines ν_3 which can be shown to be

$$\nu_3 = -\frac{k}{I_a\omega_1^2}[\Gamma\{(k_3/k_1)(k^2+n^2-k^2K_a^2)+r_{31}kK_a-s_{31}\}+q_{31a}+2k_3k^2/k_1]-p_{31}+\frac{k_3}{k_1}\frac{k}{I_a}\left\{(n^2+k^2)\frac{I_a^2}{k^2}-1\right\}-\frac{k}{I_a}q_{31b}. \quad (5.11)$$

Equation (5.11) shows that ν_3 is infinite when $\omega_1=0$ at $k=k_1$. Thus, for ν_3 to be finite at $\omega_1=0$, we equate to zero the quantity in the bracket of (5.11) when $k=k_1$. Thus

$$k_3 = -k_1\frac{\Gamma(r_{31}k_1K_a-s_{31})+q_{31a}}{\Gamma(k_1^2+n^2-k_1^2K_a^2)+2k_1^2}. \quad (5.12)$$

6 Numerical results and discussion

When the initial amplitude η_0 is plotted against the numerically obtained break-up time T_B on a semi-logarithmic scale, a linear relationship emerges, showing that the temporal growth of the disturbance is exponential-like, just as in the linear theory.

In the numerical examples in the following, n is chosen to be 1. Figures 2 and 3 show this exponential character for various initial amplitudes. The slopes of these lines can be used to calculate the actual growth rates. If we examine these figures we can notice that, with the same initial amplitude, it takes more time for the breakup of

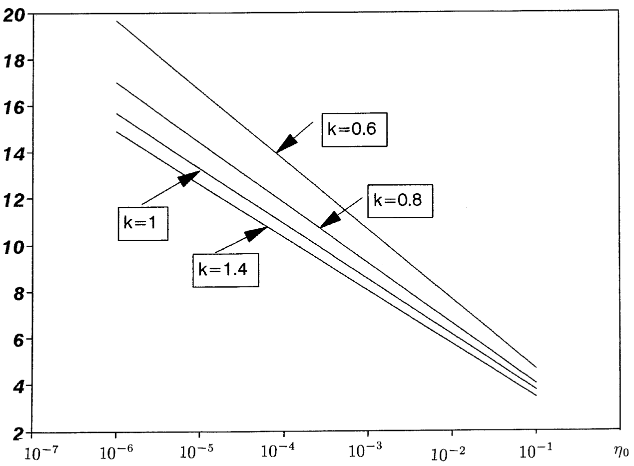

Fig. 3. Exponential character of the growth of surfaces waves ($n = 1, \Gamma = 2.5$).

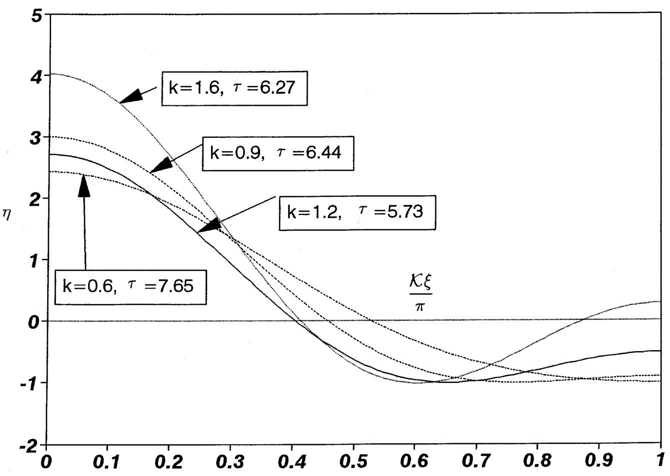

Fig. 5. Wave profiles at the breakup for the dimensionless wave number $\Gamma = 2.5, \eta_0 = 0.01$.

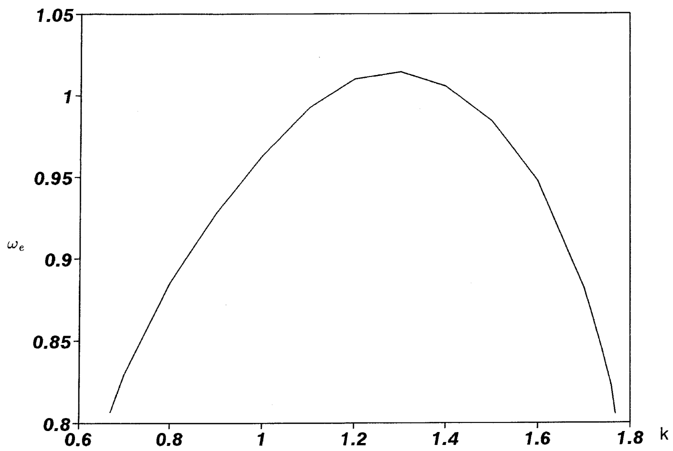

Fig. 4. Effective growth factor ω_e plotted as a function of the wave number.

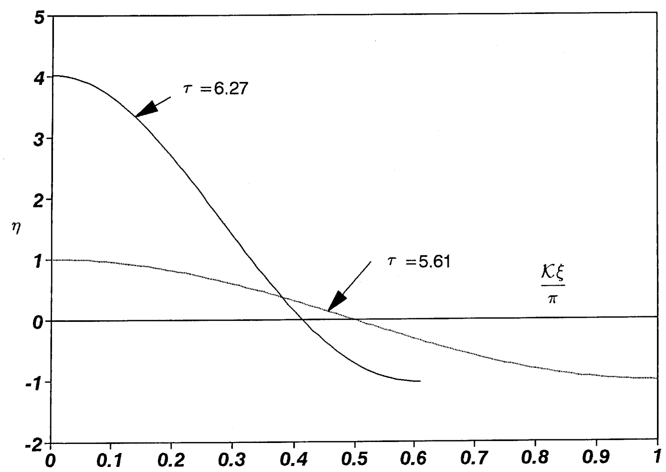

Fig. 6. Comparison of the linear ($\cdots$) and nonlinear wave profiles at the breakup. $k = 1.6$.

the jet when k is greater. This linear relationship between breakup times and the logarithms of amplitude is

$$\frac{\ln \eta_0 - \ln \eta_0'}{T_B - T_B'} = \omega_e,$$

where ω_e is the effective growth rate and T_B is the breakup time. ω_e was calculated and plotted as a function of the wave-number in Figure 3. Figure 4 shows that when Γ=2.5 the curve resembles a parabola which attains its maximum value at about $k = 1.3$.

In addition to the fundamental mode, the presence of various harmonics in $\eta(\theta, \xi, \tau)$ is the result of the energy transfer from the fundamental mode to the modes of higher orders. The breakup time of the jet is obtained by increasing the time τ in η until the deepest trough of the wave coincides with the centerline of the jet. The breakup time and breakup point was determined by increasing τ and ξ simultaneously until $\eta(\theta, \xi, \tau) = -1$ is attained. With the critical value so obtained, $\eta(\theta, \xi, \tau)$ is used to plot the curves for η *versus* ξ for various values of k when Γ=2.5 in Figure 5.

In Figure 6 we compare the linear and nonlinear wave profiles when $k = 1.6, \eta_0 = 0.01$ and $\Gamma = 2.5$. The dotted line represents a linear curve while solid line indicates the nonlinear profile. We notice that the sinusoidal curve develops into an asymmetrical form from the higher harmonics, and the breakup time is greater with the nonlinear theory. It should also be mentioned that k_3 is not valid for all wave-numbers.

Appendix A

$$a_{21} = -(b_{21} + c_{21}), \tag{A.1}$$

$$b_{21} = \frac{1}{8I_b(\omega_{21}^2 - 4\omega_1^2)}[2\omega_1^2 I_b\{1 - 2kI_a(1 + n^2/k^2)\} + k\{\omega_1^2(3 - I_a^2(1 + n^2/k^2)) + 2 + k^2 - 7n^2\} - k\Gamma\Delta_1], \tag{A.2}$$

$$c_{21} = \frac{1}{8I_b\omega_{21}^2}[k\{\omega_1^2(1+I_a^2(1+n^2/k^2)) +2+k^2-7n^2\} - k\Gamma\Delta_1], \quad (A.3)$$

$$a_{22} = -(b_{22}+c_{22}), \quad (A.4)$$

$$b_{22} = \frac{1}{8I_0(\omega_{22}^2-4\omega_1^2)}[2\omega_1^2 I_0(1-2kI_a) + k\{\omega_1^2(3-I_a^2(1-n^2/k^2))+2+k^2-n^2\} - k\Gamma\Delta_2], \quad (A.5)$$

$$c_{22} = \frac{1}{8I_0\omega_{22}^2}[k\{\omega_1^2(1+I_a^2(1-n^2/k^2)) + 2+k^2-n^2\} - k\Gamma\Delta_2], \quad (A.6)$$

$$a_{23} = -(b_{23}+c_{23}+\frac{1}{8}), \quad (A.7)$$

$$b_{23} = \frac{1}{8(\omega_{23}^2-4\omega_1^2)}[2\omega_1^2(1-2I_a n^2/k) + n\{\omega_1^2(3+I_a^2(1-n^2/k^2)) + 2-k^2-7n^2\} - n\Gamma\Delta_3], \quad (A.8)$$

$$c_{23} = \frac{1}{8\omega_{23}^2}[n\{\omega_1^2(1-I_a^2(1-n^2/k^2)) + 2-k^2-7n^2\} - n\Gamma\Delta_3], \quad (A.9)$$

$$\Delta_1 = 2kK_b(kK_a+1) + kK_a(kK_a+4) + 3 - 3k^2 - 3n^2, \quad (A.10)$$

$$\Delta_2 = 2kK_0(kK_a+1) + kK_a(kK_a+4) + 3 - 3k^2 - n^2, \quad (A.11)$$

$$\Delta_3 = -2n(kK_a+1) + kK_a(kK_a+4) + 3 - k^2 - 3n^2, \quad (A.12)$$

$$\Delta_4 = kK_a(kK_a+4) + 4 - k^2 - n^2, \quad (A.13)$$

and

$$\omega_{21}^2 = \frac{2k}{I_b}\{1-4k^2-4n^2-\Gamma(2kK_a+1)\}, \quad (A.14)$$

$$\omega_{22}^2 = \frac{2k}{I_0}\{1-4k^2-\Gamma(2kK_0+1)\}, \quad (A.15)$$

$$\omega_{23}^2 = 2n\{1-4n^2-\Gamma(1-2n)\}, \quad (A.16)$$

with

$$I_b = \frac{I_{2n}(2k)}{I'_{2n}(2k)}, \quad K_b = \frac{K'_{2n}(2k)}{K_{2n}(2k)}, \quad I_0 = \frac{I_0(2k)}{I_1(2k)}, \quad K_0 = -\frac{K_1(2k)}{K_0(2k)}\cdot \quad (A.17a–d)$$

Appendix B

$$P_{31}(\tau) = p_{31}\omega_1 \sinh\omega_1\tau, \quad Q_{31}(\tau) = q_{31}\cosh\omega_1\tau$$

$$R_{31}(\tau) = r_{31}\cosh\omega_1\tau, \quad S_{31}(\tau) = s_{31}\cosh\omega_1\tau$$

$$\alpha_1 = \frac{b_{21}}{2}+c_{21}, \quad \alpha_2 = b_{22}+2c_{22}, \quad \alpha_3 = b_{23}+2c_{23},$$

$$\beta_1 = b_{21}+\frac{P_{21}}{2\omega_1}, \quad \beta_2 = 2b_{22}+\frac{P_{22}}{\omega_1}, \quad \beta_3 = 2b_{23}+\frac{P_{23}}{\omega_1},$$

$$\gamma_1 = \frac{b_{21}}{2}-c_{21}, \quad \gamma_2 = b_{22}-2c_{22}, \quad \gamma_3 = b_{23}-2c_{23}$$

$$p_{31} = -\frac{1}{4k}\left[\beta_1 I_b\frac{k^2+n^2}{k} + \beta_2 I_0 k + \beta_3 n - (\beta_1+\beta_2+\beta_3)\right] - \frac{I_a}{2k}[\gamma_1(k^2+n^2)+\gamma_2 k^2+\gamma_3 n^2] - \frac{3I_a}{128k}[k(k^2+n^2+6) - I_a(5n^2+3k^2)] + \frac{I_a}{4k}\{I_a(k^2+n^2)-k\}\left(\gamma_1+\gamma_2+\gamma_3+\frac{1}{8}\right), \quad (B.1)$$

$$q_{31} = q_{31a} + \frac{2k_3k^2}{k_1} + \omega_1^2\left(q_{31b}+\frac{\nu_3}{k}I_a\right), \quad (B.2)$$

$$q_{31a} = -\frac{1}{2}\{\alpha_1(k^2+n^2)+\alpha_2 k^2+\alpha_3 n^2\} + \frac{9}{64}\left(\frac{3n^2+k^2}{2}-3\right) + \frac{1}{2}\left(\alpha_1+\alpha_2+\alpha_3-\frac{3}{8}\right)(1-n^2) + \frac{27}{64}n^2 - \frac{3}{128}\{2k^2n^2+9(k^4+n^4)\}, \quad (B.3)$$

$$q_{31b} = \frac{1}{2}\left[\{I_a(k^2+n^2)-k\}\frac{45}{64k}+\beta_1+\beta_2+\beta_3\right] + \frac{3I_a}{64k}\left(n^2+k^2-\frac{I_a}{k}n^2\right) + \frac{I_a}{4k}\{\beta_1 I_b\frac{k^2+n^2}{k}+\beta_2 I_0 k+\beta_3 n\} + \frac{1}{4}\left(\alpha_1+\alpha_2+\alpha_3-\frac{3}{8}\right), \quad (B.4)$$

$$r_{31} = -\frac{1}{2}\left[kK_b\alpha_1 + kK_0\alpha_2 - n\alpha_3 - \frac{3}{16}\left(\frac{kK_b}{2}+kK_0-n\right)(2kK_a+1)\right] - \frac{kK_a}{4}\left(\alpha_1+\alpha_2+\alpha_3-\frac{3}{8}\right) - \frac{27}{128}(k^2+n^2-kK_a) + \frac{9}{64}, \quad (B.5)$$

$$
\begin{aligned}
s_{31} = &\frac{1}{4}(\alpha_1+\alpha_2+\alpha_3-\frac{3}{8})(3-k^2-n^2+2kK_a)\\
&+\frac{27}{128}\{kK_a(k^2+n^2-2kK_a-6)+3k^2+5n^2\}\\
&+\frac{1}{2}\Bigg[kK_b\alpha_1+kK_0\alpha_2-n\alpha_3\\
&-\frac{3}{16}\left(\frac{kK_b}{2}+kK_0-n\right)(2kK_a+1)\Bigg](kK_a+2)\\
&-\frac{1}{2}\{(n^2+k^2)\alpha_1+k^2\alpha_2+n^2\alpha_3\}\\
&+\frac{9}{64}(n^2+k^2)(3kK_a+1)\\
&-\frac{9}{64}(6+n^2), \qquad \text{(B.6)}
\end{aligned}
$$

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